MATH 5061 Solution to Problem Set 4[1]

1. Let (M,g) be a Riemannian manifold. Fix $p \in M$.

   (a) Suppose the exponential map $\exp_p$ is defined on the whole tangent space T_pM. Prove that for any $q \in M$, there exists a geodesic γ joining p to q such that $L(\gamma)$ realized the Riemannian distance $\rho(p,q)$ between p and q. Use this to show that (M,ρ) is complete as a metric space.

   (b) Prove the converse of (a), i.e. suppose (M,ρ) is a complete metric space, show that $\exp_p$ is well-defined on T_pM.

Solution:

(a). Let $B_\delta(p)$ be a small ball centered at p such that for any $s_1, s_2 \in \overline{B_\delta(p)}$, there is a unique geodesic γ jointing s_1, s_2 such that $\rho(s_1,s_2) =$ Length of γ. We call this ball as the totally normal ball at p. Let $S_\delta(p)$ be the boundary of $B_\delta(p)$. Note that $S_\delta(p)$ is a compact set, we can find some $s \in S_\delta(p)$ such that $\rho(s,q)$ attains a minimum on $S_\delta(p)$. So we can find a minimizing geodesic $\gamma(t)$ with $\gamma(0)=p, \gamma(\delta)=s$ and $\|\gamma'(0)\|=1$. By the definition of exponential map, we have $\exp_p(\delta v)=s$ where $v=\gamma'(0)\in T_pM$. Let $l=\rho(p,q)$, we are going to show $\exp_p(lv)=q$. Since $\exp_p(tv)$ defined for all $t\in\mathbb{R}$, we actually can extend the definition of $\gamma(t)$ for $t\in\mathbb{R}$ by $\gamma(t)=\exp_p(tv)$.

We consider the following equation.

$$\rho(\gamma(t),q) = l - t \tag{1}$$

Let $A=\{t\in(0,l]:\ (\textbf{??})\text{ holds for } t\}$. Clearly $A\neq\emptyset$ since $\delta\in A$ (Triangle inequality $\implies \rho(s,q)\geq l-\delta$. If $\rho(s,q)=l_0>l-\delta$, then any piecewise smooth curve jointing p,q will $\geq l_0+\delta$ since they will pass through $S_\delta(p)$.)

Note that A is closed in $(0,l]$ by the continuous of distance. So let's show if $t_0\in A$ and $t_0\neq l$, then we can find $\delta'>0$ such that $t_0+\delta'\in A$. Still we choose a totally normal ball $B_{\delta'}(\gamma(t_0))$ such that $p,q\notin B_{\delta'}(\gamma(t_0))$. So we know $\delta'\leq\rho(\gamma(t_0),q)=l-t_0 \implies t_0+\delta'\leq l$. Again, we can find some $s'\in S_{\delta'}(\gamma(t_0))$ such that $\rho(s,q)$ attains a minimum on $S_{\delta'}(\gamma(t_0))$. We claim $s'=\gamma(t_0+\delta')$. If not, we note $\rho(s',\gamma(t_0-\delta'))<\rho(s',\gamma(t_0))+\rho(\gamma(t_0),\gamma(t_0-\delta'))=2\delta'$ by the definition of totally normal ball. Hence $\rho(s',p)<t_0+\delta'$. Again by triangle inequality, $\rho(q,s')\geq l-\rho(x',p)>l-t_0-\delta'$. Since any curves jointing $\gamma(t_0),q$ will pass through $S_{\delta'}(\gamma(t_0))$, we actually know $\rho(q,\gamma(t_0))\geq\rho(x',q)+\delta'>l-t_0$, a contradiction with $t_0\in A$. So we should have $x'=\gamma(t_0+\delta')$. Still by triangle inequality $\rho(\gamma(t_0+\delta'),p)\geq l-t_0-\delta'$ but $\rho(\gamma(t_0+\delta'),p)>l-t_0-\delta'$ cannot hold by the same reason. Hence $\rho(\gamma(t_0+\delta'),q)=l-t_0-\delta' \implies t_0+\delta'\in A$.

The above steps show $\sup A\in A$ by the closeness and moreover $\sup A=l$. Hence $l\in A$ and $\gamma(l)=q$. The γ is the geodesic jointing p,q realized the distance $\rho(p,q)$.

To prove (M,ρ) is complete, note for any Cauchy sequence (p_i), we know $\rho(p_i,p)$ is bounded by Triangle inequality. Suppose $\rho(p_i,p)<M$ for all i, we know p_i in the image of $\overline{B_M(p)}$ under the map $\exp_p$. Note $\overline{B_M(p)}$ is compact, so does the set $\exp_p(\overline{B_M(p)})$. Hence we can find a convergent subsequence of (p_i) and indeed the whole sequence will have the same limit since it is Cauchy.

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(b). Let's suppose $\exp_p$ is not defined on the whole T_pM. That means there is a geodesic $\gamma(t)$ with $\gamma(0) = p$ is not defined for some t. WLOG, we assume $\|\gamma'(0)\| = 1$. By the existence of geodesic, we know there is a largest open interval $(-s_0, s_1)$ such that $\gamma(t)$ is well-defined. Let $t_i \in (-s_0, s_1)$ such that $t_i \to s_1$. Note $\rho(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|$, $\gamma(t_i)$ is Cauchy and we can find $q \in M$ such that $\gamma(t_i) \to q$.

Now let $B_\delta(q)$ be a totally normal ball at q. We can find N large such that $p_i \in B_{\frac{\delta}{2}}(q)$ and $|t_i - s_1| < \frac{\delta}{2}$ for all $i > N$. Note that any two points in $B_\delta(p)$ can be joined by a minimizing geodesic, we know the exponential map $\exp_{p_i}$ defined for all $\|v\| \leq \frac{\delta}{2}$. Let's consider two points p_i, p_j with $N < i < j$ and they're joined by a minimizing geodesic $\gamma(t), t \in [t_i, t_j]$. But note $\exp_{t_j}(t\gamma'(t_j))$ exists for $t \in [-\frac{\delta}{2}, \frac{\delta}{2}]$, we know $\gamma(t)$ is well-defined when $t \in [t_j, t_j + \frac{\delta 2}{2}]$. Note $t_j + \frac{\delta}{2} > s_1$, so it contradicts with the choice of γ. Hence $\exp_p$ is defined on the whole T_pM.

2. Prove that every Jacobi field V along a geodesic γ in (M, g) arises from the variation vector field of a 1-parameter family of geodesics.

<u>**Solution:**</u>

Suppose $\gamma(t)$ defined on $[0, T]$ with $\|\gamma'(0)\| = 1$. Let $\tilde{\gamma}(s) : (-\varepsilon, \varepsilon) \to M$ be the geodesic starting from $\gamma(0)$ with initial velocity $V(0)$. So we consider the variation of γ defined by

$$f(t, s) = \exp_{\tilde{\gamma}(s)}(tW(s))$$

where $W(s)$ be the vector field along $\tilde{\gamma}$ with $W(0) = \gamma'(0)$ and $\frac{DW}{ds}(0) = \frac{DV}{dt}(0)$.

Clearly $f(t, 0) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t)$, so f is indeed a variation of γ.

Note that the variation of geodesic will give the Jacobi field. That is, if we define $\tilde{V}(t) = \frac{\partial f}{\partial s}(t, 0)$, the vector field along γ, then note $\frac{D}{dt}\frac{\partial f}{\partial t} = 0$, we have

$$\begin{aligned} 0 &= \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} - R(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t})\frac{\partial f}{\partial t} \\ &= \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s})\frac{\partial f}{\partial t} = \nabla_{\gamma'}\nabla_{\gamma'}\tilde{V} + R(\gamma', \tilde{V})\gamma' \end{aligned}$$

which shows $\tilde{V}$ is indeed a Jacobi field.

Note that $V(0) = \tilde{V}(0) = \frac{\partial f}{\partial s}(0, 0)$ and

$$\nabla_{\gamma'(0)}\tilde{V}(0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0, 0) = \frac{D}{ds}W(0) = \frac{DV}{dt}(0) = \nabla_{\gamma'(0)}V(0)$$

Hence $V = \tilde{V}$ along γ by the uniqueness of the ODE solutions.

So V arises as the variation of the geodesic.

3. A vector field $X \in \Gamma(TM)$ is said to be a *Killing vector field* if $\mathcal{L}_X g = 0$.

   (a) Suppose M is compact. Show that X is a Killing vector field if and only if the flow $\{\varphi_t\}$ of diffeomorphisms of M generated by X consists of isometries of (M, g).

   (b) Prove that any Killing vector field X restricts to a Jacobi field on every geodesic in M.

   (c) Suppose M is connected. Show that a Killing vector field X on M which vanishes at some $p \in M$ and $\nabla_Y X(p) = 0$ for all $Y(p) \in T_pM$ must vanish everywhere on M.

Solution:

(a).

" $\Longrightarrow$ " Suppose X is a Killing vector field. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_t^* g = \mathcal{L}_X g = 0$$

where φ_t generated by X by the definition of Lie derivative of tensor. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0}\varphi_t^* g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_{t_0}^* \circ \varphi_t^* g = \varphi_{t_0}^* \left(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_t^* g\right) = \varphi_{t_0}^*(\mathcal{L}_X g) = 0$$

by the properties of flow. So $\varphi_t^* g = g$.

" $\Longleftarrow$ " Suppose $\varphi_t^* g = g$ for all t, then

$$\mathcal{L}_X g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi_t^* g = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} g = 0$$

(b). Let φ_t be the flow generated by X. So φ_t will be the isometries of M. Hence for any geodesic $\gamma(s)$, the variation of γ defined by $\gamma_t(s) = \varphi_t(\gamma(s))$ is geodesic for every $t \in \mathbb{R}$. Hence the vector field $V = \frac{\partial \gamma_t}{\partial t} = \frac{\partial}{\partial t}\varphi_t = X$ along $\gamma(s)$ is a Jacobi field.

(c). Let $A = \{p \in M : X, \nabla_Y X \text{ vanished at } p \text{ for all } Y(p) \in T_pM\}$. Clearly $A \neq \emptyset$ is closed. We show A is open, too. For any $p \in M$, we choose a small ball $B_\delta(p)$ that for every point in q, there is a unique minimizing geodesic $\gamma_{p,q}$ in $B_\delta(p)$ jointing p, q. Note that X is a Jacobi field along $\gamma_{p,q}$ that $X, \nabla_{\gamma_{p,q}'(0)}$ vanish at p. But by the uniqueness of Jacobi field when given $V(0), \nabla_{\gamma'(0)} V(0)$, we know X should be the zero vector field. Hence X will be zero in the whole ball $B_\delta(p)$. So $B_\delta(p) \subset A$.

Since M is connect and A is open and closed at the same time, we know $A = M$.

4. Show that Synge theorem does not hold in odd dimensions.

Solution:

Consider the projective space $\mathbb{RP}^n$ when n is odd. It is a quotient of $\mathbb{S}^n$ under antipodal map φ. Since φ is orientation preserving when n is odd, we know $\mathbb{RP}^n$ is orientable and moreover $\mathbb{RP}^n$ has positive sectional curvature by the properties of covering map. Hence Synge theorem does not hold in odd dimensions.